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Donaldson–Witten Invariants and Pure 4D-QCD with Order and Disorder 't Hooft-like Operators

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Abstract

We study the first-order formalism of pure four-dimensional SU(2) Yang–Mills theory with theta-term. We describe the Green functions associated to electric and magnetic flux operators à la 't Hooft by means of gauge-invariant non-local operators. These Green functions are related to Witten's invariants of four-manifolds.

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1 Introduction

Seiberg and Witten [1] studied recently the strongly coupled infrared limit of minimal $N = 2$ super Yang–Mills theory. The key point in their work is the existence of an electric-magnetic S-duality [2, 3, 4], which extends the Olive–Montonen duality [6]. By taking into account the low energy effective theory, Witten [7] introduced an interesting "monopole equation"[5], describing abelian gauge fields weakly coupled to monopoles. By counting the solutions of the "dual" equation, Witten obtains new invariants for a compact, simply connected, oriented 4-manifold with a distinguished integral cohomology class. These invariants are closely related to Donaldson polynomial invariants.

On the other hand, the infrared phase of the $N = 2$ super Yang–Mills theory is like 't Hooft's "superinsulator" (dual superconductor), where one gets absolute confinement of electric charges by the monopole-condensation mechanism. This phase is controlled by a set of topological operators, implicitly defined by 't Hooft, that create or destroy topological quantum numbers. We call such operators "disorder operators". 't Hooft argued that the correlators of the disorder operators are a sort of "topological" Green's function.

In [9, 10, 11] the authors consider three- and four-dimensional BF-theory and define new gauge-invariant, non-local operators $M(\Sigma)$ associated to surfaces Σ immersed or embedded in a four-dimensional manifold.

It turns out that these operators $M(\Sigma)$ are an explicit realization of 't Hooft disorder operators.

In this work we generalize the ideas of [9, 11, 12] and prove that the first-order formalism of the standard pure bosonic Yang–Mills theory with theta-term (which is a special type of BF-theory) allows us to get an explicit realization of the 't Hooft scenario.

Moreover we show that the expectation values of the non-local operators $M(\Sigma)$ in the weak coupling regime provide solutions of a "monopole equation".

When we consider a Kähler manifold, then our monopole equation is related to the one considered by Witten, provided that some extra relations are fulfilled [7] (see (36)).

The inclusion in the expectation values of $M(\Sigma)$ of the "order parameters" provided by Wilson operators "perturbs" the above topological Green functions. The new correlation functions appear to be related to Donaldson invariants.

We would like to stress that the main ingredient here is the fact that monopole equations select singular abelian connections. In our picture the "source" of the monopole equation is provided by a bosonic 2-form, which appears naturally in the first order formalism of pure Yang–Mills theory, while in the $N = 2$ super Yang–Mills theory [7] such a source is given by a bilinear form of a complex Weyl spinor.

Our approach may give a hint on how to perform a completely non-perturbative calculation directly in standard 4D-QCD, without the need to get it by some spontaneous

[5] A monopole equation can be obtained also in the framework of the twisted $N = 2$ sigma-model [8].

symmetry breaking of an $N=2$ super Yang–Mills theory [1]. Our result gives more support to the general intuition that in standard 4D Yang–Mills theory, there should be a topological sector controlled by the topological Green's functions of suitable gauge-invariant, non-local, composite operators.

2 4D-QCD in first order formalism

Let us start with some mathematical definitions. Let X be a compact, oriented 4-dimensional manifold, and more specifically a Kähler manifold with Kähler form ω (i.e. ω is a closed $(1,1)$-form). On X we consider a non-abelian gauge theory. From a mathematical point of view this amounts to defining a complex vector bundle E associated to a principal bundle with structure group G. We will always consider the case of $G = \mathrm{SU}(2)$. We will write $\mathfrak{g}_E$ for the bundle associated to the adjoint representation of SU(2). If A denotes a unitary connection on E, then we denote with $d_A : \Omega^p_X(E) \to \Omega^{p+1}_X(E)$ the associated covariant exterior derivative. Notice that $d_A = d+A$ on $\Omega^0_X(E)$. Here A stands for an $\mathfrak{su}(2)$-valued one-form which transforms locally as $A \to A_u \equiv u^{-1}Au+u^{-1}du$, where u is (a local representative of) an element of the gauge group, i.e. a smooth map $u : X \to \mathrm{SU}(2)$, and $\Omega^p_X(E)$ denotes as usual the space of p-forms on X with values in E. The Yang–Mills curvature, i.e. the curvature of the connection A, is the $\mathfrak{su}(2)$-valued two-form $F_A = dA+A\wedge A \in \Omega^2_X(\mathfrak{g}_E)$ which transforms as $F_A \to F_{A_u} \equiv u^{-1}F_A u$. Notice that for any connection A we have the Bianchi identities $d_A F_A = 0$. If we fix a metric structure on X and denote by $*$ the associated Hodge operator we have the splitting into self-dual and anti-self-dual parts of the bundle-valued 2-form

$$F_A = F_A^+ \oplus F_A^- \in \Omega^{2,+}_X(\mathfrak{g}_E) \oplus \Omega^{2,-}_X(\mathfrak{g}_E) \tag{1}$$

We call the connection A anti-self-dual (ASD) if $F_A^+ \equiv \left(\frac{1+*}{2}\right) F_A = 0$. Since X is by hypothesis a Hermitian manifold we can also decompose the curvature F_A as

$$F_A = F_A^{(2,0)} \oplus F_A^{(1,1)} \oplus F_A^{(0,2)} \in \Omega^{(2,0)}_X(\mathfrak{g}_E) \oplus \Omega^{(1,1)}_X(\mathfrak{g}_E) \otimes \Omega^{(0,2)}_X(\mathfrak{g}_E). \tag{2}$$

Now one can show that the self-dual bundle-valued complexified 2-forms over X are [14]:

$$\Omega^{2,+}_X(\mathfrak{g}_E) = \Omega^{(2,0)}_X(\mathfrak{g}_E) \oplus \omega\Omega^0_X(\mathfrak{g}_E) \oplus \Omega^{(0,2)}_X(\mathfrak{g}_E) \tag{3}$$

with ω the Kähler $(1,1)$-form ($\omega\Omega^0_X(\mathfrak{g}_E) = \Omega^{(1,1)}_X(\mathfrak{g}_E)$). In (1-3) the $\Omega^{(p.q)}_X(\mathfrak{g}_E)$ denote the sections $\Gamma(\Lambda^{(p,q)}_X \otimes \mathfrak{g}_E)$ and the $\Omega^{\pm}_X$ stand for the algebraic projections $\frac{1}{2}(1\pm *)\Omega_X \equiv \pi_\pm \Omega_X$.

Our physical theory is the first-order version of the pure SU(2) Yang–Mills theory with the so-called "theta-term" given by the action functional (in Euclidean signature)

$$S_{BF^+} = \int_X \mathrm{Tr}(B\wedge F_A^+) - \frac{g^2}{4}\int_X \mathrm{Tr}(B\wedge B), \tag{4}$$

$B \in \Omega^2_X(\mathfrak{g}_E)$, and hence $B \to B_u \equiv u^{-1}Bu$ under gauge transformations. In (4), g^2 denotes the bare gauge coupling constant. The field equations obtained from (4) are

$$F_A^+ = \frac{g^2}{2}B, \tag{5}$$

$$(d_A^+)^* B = 0 \tag{6}$$

where the + superscript denotes the self-dual projection and $d_A^* : \Omega_X^{p+1}(E) \to \Omega_X^p(E)$ is the adjoint operator of the covariant derivative. Eq. (5) clearly implies that $B \in \Omega^{2,+}$ and hence one may set $B = B^+$. In the following we shall denote by $\mathcal{M}_c$ the "classical" moduli space, that is the space of solutions of (5-6) modulo the gauge group. Now using (5) in (4) one gets the action functional in the second-order formalism (in the Euclidean signature):

$$S_{\mathrm{YM}} = \frac{1}{g^2}\int_X \mathrm{Tr}(F_A^+ \wedge F_A^+) = \frac{1}{2g^2}\int_X \mathrm{Tr}(F_A \wedge *F_A) + \frac{1}{2g^2}\int_X \mathrm{Tr}(F_A \wedge F_A). \tag{7}$$

At this point one may reabsorb the gauge coupling g into F_A through the rescaling $A \to A' = A/g$, so that (5) now reads

$$F'^+_{A'} \equiv (dA' + gA' \wedge A')^+ = \frac{g}{2}B. \tag{8}$$

Notice that the first term in the right-hand-side of (7) is the standard Yang–Mills functional, while the second one is the topological charge, the so-called "theta-term" associated with the instanton number k [14]

$$k(A) = -c_2(E)[X] = -\frac{1}{8\pi^2}\int_X \mathrm{Tr}(F_A \wedge F_A). \tag{9}$$

Here $c_2(E)$ is the second Chern class of the complex SU(2)-vector bundle E and we have adopted the standard sign convention for k. If we denote by t_a the generators of $\mathfrak{su}(2)$ (in the fundamental representation) we can write $A = \sum_a A^a_\mu t_a dx^\mu$, $F_A = \sum_a (F_A)^a_{\mu\nu} t_a dx^\mu \bigwedge dx^\nu$ in a local coordinate system $\{x^\mu\}$ on X. Furthermore, we fix the normalization for the t_a's according to $\mathrm{Tr}(t_a t_b) = -\frac{1}{2}\delta^{ab}$. For SU(2) we get that $t_a = i\sigma_a/2$ $(a = 1, 2, 3)$ with σ^a the Pauli matrices.

In the following we shall consider the quantum theory associated to the action (4) in the frame of path-integral quantization. For this purpose we need to fix the gauge-invariance of (4). We use the background covariant "gauge fixing" condition

$$d^*_{A_0} a = 0 = d^*_{A_0} B, \tag{10}$$

where we have defined $A = A_0 + a$ with A_0 a fixed background SU(2)-connection. Notice that we have to impose a constraint on the field B, since the theory acquires on shell a larger symmetry. We will stick to the common terminology and call this constraint on the field B a "gauge fixing". Since (4) is just the 4-dimensional Yang–Mills theory (with theta-term) [6] written in the first-order formalism, and we know that a non-abelian gauge theory is

[6] In Euclidean signature the Gibbs measure associated to the "theta action-functional"is usually written as $\exp(i\theta k(A))$, where $k(A)$ is defined by (9) and the Lagrange multiplier θ is the so-called θ-parameter. Then (7) implies $\theta = 4\pi^2$, which is consistent with the fact that θ does not renormalize.

asymptotically free, it follows that the B field is weakly (strongly) coupled to the Yang–Mills potential A in the ultraviolet (infrared) regime. Therefore (8), after the replacing of g with the running coupling constant $g(\mu)$ ($g(\mu) \sim 0$ at energies $\mu \gg M$ with respect to some physical energy scale M) becomes the ASD-condition $F'^{+}_{A'} \sim 0$ in the ultraviolet regime.

An alternative theory which, in the semiclassical approximation, is equivalent to the one described by (4) is given by the following topological action

$$S_{BF} = \int_X \mathrm{Tr}(B \wedge F_A) - \frac{g^2}{4} \int_X \mathrm{Tr}(B \wedge B), \tag{11}$$

provided that we assume:

$$B^- = 0. \tag{12}$$

By assuming the above constraint, we are breaking the "topological" symmetries of the BF theory (11) given by:

$$\delta_t A = \frac{g^2}{2}\eta \tag{13}$$
$$\delta_t B = -d_A \eta, \tag{14}$$

where η is a 1-form ghost [9][11].

3 Quantum theory

The quantum theory associated to (4) is determined by the functional integral

$$\langle 1 \rangle_{\mathcal{M}} = \int_{\mathcal{M}} d\mu(A,B) e^{-S_{BF^+}}. \tag{15}$$

In (15) $\mathcal{M}$ is the orbit space, i.e. the quotient of the set of fields (A, B) by the gauge group. Following the usual procedure, the functional measure $d\mu(A, B)$ over $\mathcal{M}$ is obtained by the standard Faddeev-Popov procedure

$$d\mu(A,B)|_{A=A_0+a} = [da][dB]\delta(d^*_{A_0} a)\delta(d^*_{A_0} B) J(A,B), \tag{16}$$

with $a \in \Omega^1(\mathfrak{g}_E)$ and the functional Jacobian $J(A, B) = \det'[\delta_\mu(d^*_{A_0} a_u)] \det'[\delta(d^*_{A_0} B_u)]$ is the standard Faddeev-Popov determinant. Of course, in order to derive (16) in the background gauge one assumes that the topology of X allows only irreducible connections A on the SU(2)-bundle E over X. This means that there are no covariantly constant scalar fields. A sufficient condition for that [14] is that $b_2^+ > 1$; here b_2^+ is the dimension of the space of self-dual harmonic forms over X.

Our key idea is to include into (15) some gauge-invariant non-local operator $M(\Sigma)$ (to be defined in the next section), which is analogous to the disorder loop variable of 't Hooft [15], in such a way that one has

$$\langle M(\Sigma) \rangle_{\mathcal{M}} = \int_{\tilde{\mathcal{M}}} d\tilde{\mu}_{L_\Sigma}(\tilde{A}, \tilde{B}) e^{-\tilde{S}_{BF^+}} \equiv \langle 1 \rangle_{(\tilde{\mathcal{M}}, L_\Sigma)} \tag{17}$$

where $\tilde{\mathcal{M}}$ is a particular case (namely the 0-dimensional one) of the Witten moduli space which gives the new differential-topological invariants for compact, 4-manifolds recently discovered by Witten [7]. Eq. (17) is one of our basic results. In the following we shall define explicitly in terms of (A, B) a non-local gauge invariant operator $M(\Sigma)$ associated to a surface $\Sigma \subset X$, such that(17) holds identically, in the semiclassical approximation and in the weak coupling limit (up to some irrelevant normalization). In (17) the symbols have the following meaning: the introduction of the observable $M(\Sigma)$ brings into the picture (singular) reducible connections that are not allowed in the theory without such observables. In other words we obtain (abelian) connections (denote by the symbol $\tilde{A}$) on the holomorphic line bundle L_Σ over X having first Chern class $c_1(L_\Sigma)$ equal to the Kähler class $[\omega] \in H^2(X)$ which is the Poincaré dual (PD) of [the homology class of] the surface $\Sigma \subset X$, i.e. $[\Sigma] = \mathrm{PD}([\omega]) = \mathrm{PD}(c_1(L_\Sigma))$[7]. $\tilde{B}$ is the 2-form B restricted to a generic $(1,1)$-section of the bundle $\Lambda_X^{(1,1),+} \otimes L_\Sigma$ over X. Notice that the condition that Σ be Poincaré dual to the Kähler form ω means that there is a singular 1-form η on X such that for any 2-form $\hat{\theta}$ over X one has

$$\Sigma = \mathrm{PD}(\omega) \iff \int_{\Sigma \subset X} \hat{\theta} - \int_X \hat{\theta} \wedge \omega = \int_X d\hat{\theta} \wedge \eta. \tag{18}$$

In (17) the action functional $\tilde{S} \equiv S_{BF^+}(\tilde{A}, \tilde{B}; L_\Sigma; X)$ is the restriction of (4) to the fields $(\tilde{A}, \tilde{B})$ defined above. Equivalently one can say that the introduction of the operator $M(\Sigma)$ into the expectation value (15) gives an *effective* gauge theory which is controlled by a split SU(2)-bundle $E_\Sigma = L_\Sigma \oplus L_\Sigma^{-1}$, where $\Sigma \to L_\Sigma$ by the correspondence $\Sigma = \mathrm{PD}(\omega) = \mathrm{PD}(c_1(L_\Sigma))$.

4 't Hooft-like disorder operators

In a four-dimensional BF-theory, we can define as in ref. [9], a non-local gauge invariant operator associated to any closed oriented immersed surface Σ:

$$\begin{aligned} O(A, B; \Sigma, \gamma, \gamma', \bar{x}) &\equiv 2\pi q_m g \int_\Sigma \mathrm{Tr}_R \left[\mathrm{Hol}_{\bar{x}}^{y}(\gamma) B(y) \mathrm{Hol}_{y}^{\bar{x}}(\gamma') \right] \\ B(y) &\equiv \sum_a B^a(y) R_a \equiv \sum_a B^a_{\mu\nu}(y) R^a dy^\mu \wedge dy^\nu \end{aligned} \tag{19}$$

where q_m will play the rôle of a "magnetic charge" (as it will be clear after (42)), $\bar{x}$ is a base point on Σ (equivalently, one may remove the point $\bar{x}$ from Σ), R^a is an irreducible representation[8] of SU(2), Tr_R denotes the trace over R and $\gamma' \cup \gamma$ is any closed path C with base point $\bar{x}$ and passing through y.

Furthermore, $\mathrm{Hol}_{\bar{x}}^{y}(\gamma)$ in (19) stands for the holonomy of the SU(2)-connection A along

[7]From now on we will drop the square parentheses denoting (co)homology classes.
[8]We will consider later on, only the fundamental representation.

the path γ from $\bar{x}$ to y:

$$\mathrm{Hol}_{\bar{x}}^{y}(\gamma) \equiv P\exp(\int_{\bar{x}}^{y} A(x)) \tag{20}$$

It has been shown in [9, 11] that the choice of the base point $\bar{x}$ is irrelevant[9], and underlying a choice for the closed path C, one can finally set in this context:

$$M(\Sigma) \equiv \exp[O_+(\Sigma)], \qquad O_+(\Sigma) \equiv O[A, B = B^+; \Sigma, C, \bar{x}], \tag{21}$$

where O is defined by (19). Here of course O_+ is the projection by $\frac{1}{2}(1+*)$ of the integrand of O to the self-dual one. We also have to consider the standard Wilson loop observable

$$W(C) \equiv \mathrm{Tr}_R \mathrm{Hol}(C) \tag{22}$$

where C is a closed loop.

Our next step is to compute the semiclassical approximation of $\langle M(\Sigma)\rangle_{\mathcal{M}}$. We now consider a background *quasi-ASD connection* A_0, namely a connection satisfying eq.(5) and (6) with $B_0 \neq 0$ and g small enough but different from 0. The linear version of the field equations (5,6) yields the the following equations for the field fluctuations $A_0 \longrightarrow A_0 + a$ and $B_0 \longrightarrow B_0 + b$ with $(a, b) \in \Omega^1 \oplus \Omega^{2,+}$.

$$d^+_{A_0} a = \frac{g^2}{2} b \tag{23}$$

$$(d^+_{A_0})^* b = 0. \tag{24}$$

The operator t_g corresponding to (23), (24) defined by

$$\begin{aligned} t_g : \Omega^1 \oplus \Omega^{2,+} &\mapsto \Omega^1 \oplus \Omega^{2,+} \\ (a, b) &\rightarrow ((d^+_{A_0})^* b, d^+_{A_0} a - \frac{g^2}{2} b). \end{aligned} \tag{25}$$

Pure gauge fluctuations $\phi \in \Omega^0$ due to the gauge transformations of the backgrounds (A_0, B_0) are not relevant here since by choosing: $a = d_{A_0}\phi$ and $b = [B_0, \phi]$ we have that the condition $(d^+_{A_0})^*[B_0, \phi] = 0$ is compatible with the equation

$$\big(d^+_{A_0} d_{A_0}\phi, [B_0, \phi]\big) = \frac{g^2}{2} \parallel [B_0, \phi] \parallel^2$$

only if we set $\phi = 0$.

Hence the dimension of our moduli space will be simply given by the dimension of $\ker t_g$. By using the Weitzenböck formulae it is possible to show that $(a, b) \in \ker t_g$ implies $b = 0$. The same argument should also imply that $(d^+_{A_0})^* d^+_{A_0} a = 0$ holds only when $a = 0$, provided that A_0 is a quasi-ASD (not ASD) connection.

The vanishing of $\ker t_g$ can be restated by saying that the classical moduli space $\mathcal{M}_c$ is zero-dimensional or discrete. In other words the inclusion of the field B (with $g \neq 0$) seems

[9] Of course the expectation values of the operators O depend on the choice of the paths γ, γ', but this choice is irrelevant for the computation in this framework of the Donaldson-Witten invariants.

to remove the degeneracy of the instanton vacua of the classical Yang–Mills theory. We will essentially find points of the classical moduli space, given by *singular* solutions. In the semiclassical computation of $\langle M(\Sigma)\rangle_{\mathcal{M}}$ one may set $\mathcal{M} \simeq \mathcal{M}_c$.

From the point of view of the field equations, the insertion into the path-integral (15) of the operator $M(\Sigma)$ is equivalent to considering the improved gauge-fixed BF^+-action

$$S_{BF^+}(A, B, d^*_{A_0}A = 0 = d^*_{A_0}B; X) + O_+(\Sigma) \equiv S^{(\Sigma)}_{BF^+}(A, B, d^*_{A_0}A = 0 = d^*_{A_0}B; X, \Sigma \subset X) \tag{26}$$

Here $O_+(\Sigma)$ plays the rôle of a "source term" for (4). Indeed, when we make the following choice of the 2-form $\hat{\theta}$ in (18)

$$\hat{\theta}(\Sigma) \equiv 2\pi q_m g \,\mathrm{Tr}\,_R[\mathrm{Hol}^y_{\bar{x}}(\gamma) B^+(y) \mathrm{Hol}^{\bar{x}}_y(\gamma')], \qquad O_+(\Sigma) = \int_\Sigma \hat{\theta}(\Sigma) \tag{27}$$

we have:

$$\int_\Sigma \hat{\theta} = \int_X \omega \wedge \hat{\theta}, \qquad \omega \equiv \mathrm{PD}(\Sigma) \tag{28}$$

Eq. (28) follows from the fact that θ is a closed form. In fact we have:

$$\begin{aligned} d\hat{\theta}(\Sigma) &= 2\pi q_m g d \,\mathrm{Tr}\,_R[\mathrm{Hol}^y_{\bar{x}}(\gamma) B^+(y) \mathrm{Hol}^{\bar{x}}_y(\gamma')] \\ &= 2\pi q_m g \,\mathrm{Tr}\,_R[\mathrm{Hol}^y_{\bar{x}}(\gamma) d_{A_0} B^+(y) \mathrm{Hol}^{\bar{x}}_y(\gamma')] = 0 \end{aligned} \tag{29}$$

Notice that in (29) we have used the gauge condition $d^*_{A_0}B = 0$ which is the same as $*d_{A_0} * B = 0$, and implies $d_{A_0}B = 0$ since B is self-dual.

5 Moduli space and topological invariants

In this section we prove (17) in the semiclassical approximation. The starting point is given by the field equations obtained after the inclusion of a disorder operator $M(\Sigma)$ à la 't Hooft.

These field equations are obtained by applying the functional derivative $\dfrac{\delta}{\delta B^a}$ to the action $S^{(\Sigma)}_{BF^+}$ (see (26-28)) and read:

$$\begin{aligned} F^{(2,0),a}_+ &= 0 + O(g^2) = F^{(0,2),a}_+ \\ F^{(1,1),a}_+(x) &= -2\pi q_m g \omega(x) \mathrm{Tr}_R\{\mathrm{Hol}^x_{\bar{x}}(\gamma) R^a \mathrm{Hol}^{\bar{x}}_x(\gamma')\} + O(g^2) = \\ &= -2\pi q_m g \omega(x) \mathrm{Tr}_R\{\mathrm{Hol}(C_x) R^a\} + O(g^2). \end{aligned} \tag{30}$$

Here C_x is a loop based at x, $\{R^a\}$ is the basis of a representation of SU(2), $F^{(p,q)}_+$ is the (p, q)-part of F^+ and the other notation is the same as in the previous section. The holonomy is non trivial if we think that the loop C_x includes a "Dirac singularity" (see below). From now on we choose the fundamental representation of SU(2) and hence we set $R_a \equiv t_a; a = 1, 2, 3$. As a consequence of equation (30) the curvature F_A is given, in the limit $g \to 0$, by the product of a numerical constant times ω times the projection of $\mathrm{Hol}(C)$ into the Lie algebra

of $SU(2)$. This projection defines a constant direction in the Lie Algebra, so we can assume that this direction is parallel to t_3. This is equivalent to considering a (singular) reducible connection. We can furthermore choose ω as a *self-dual 2-form*. In conclusion Eq. (30) admits a non-trivial singular reducible[10] quasi ASD connection of the form [19]:

$$\tilde{A}(x) = \frac{i}{2}\begin{pmatrix} \alpha(x) & 0 \\ 0 & -\alpha(x) \end{pmatrix} = \alpha(x)t_3, \tag{31}$$

where α is a 1-form, $t_3 = \frac{i}{2}\sigma_3 = \frac{i}{2}\mathrm{diag}(1,-1)$. Hence α is an *abelian* connection and the curvature of (31) is ft_3, where we have set: $f \equiv d\alpha$. Thus the introduction of the operators $M(\Sigma)$ replaces the original non-abelian Yang–Mills theory [with theta-term] to an effective theory based on a split $SU(2)$-bundle $E_\Sigma = L_\Sigma \oplus L_\Sigma^{-1}$, with L_Σ a holomorphic line bundle on X "parameterized" by $\Sigma \subset X$, in the sense that $c_1(L_\Sigma) = \omega = \mathrm{PD}(\Sigma)$. So the instanton number $k = -c_2(E_\Sigma)$ is expressed in terms of the monopole number $\lambda = c_1(L_\Sigma)$, since one has $k = -c_2(E_\Sigma = L_\Sigma \oplus L_\Sigma^{-1}) = c_1(L_\Sigma)^2 = \lambda^2$. As a consequence of (31), we have:

$$\mathrm{Tr}(\mathrm{Hol}(C)t_a) = -\sin(g\phi_m/2)\delta_{a,3} \tag{32}$$

where ϕ_m is the monopole magnetic flux of $\tilde{F}/g$ across the "Dirac surface" Σ. One may regard the r.h.s. of (30) as an effective field $\tilde{B}$, given by

$$\tilde{B}^a = -\pi g q_m \omega \mathrm{Tr}(\mathrm{Hol}(C)t_3)\delta_{a,3} \tag{33}$$

obtained by the insertion of the operator $M(\Sigma)$ into (15) in the limits $g \to 0$ and $\hbar \to 0$. Notice that by (44) the field $\tilde{B}$ is actually independent of g and hence contributes also in the weak-coupling limit. By (30) and (33), the self dual part of the curvature of the vector potential $\tilde{A}' = \tilde{A}/g$ is given by:

$$d^+\tilde{A}' = (2/g)\tilde{B} + O(g^2) \tag{34}$$

Eqn. (34) is the same as (8) when applied to the reducible connection $\tilde{A}$, provided that we perform the so called weak-strong coupling duality $g \to 1/g$. Furthermore (24) becomes

$$(d^+_{\tilde{A}})^*\tilde{B} = 0 + O(g) \tag{35}$$

where $\tilde{B} = \tilde{B}^+$ is a $(1,1)$-section of the twisted self-dual vector bundle $\Lambda_X^{(1,1),+} \otimes L_\Sigma$ over X. Equations (34-35) are the monopole equations that, in the case of a general Kähler manifold X, correspond to Witten's monopole equations [7].

The relation between the complex Weyl spinors M and $\bar{M}$ in Witten's equation and our field $\tilde{B}$ is of the form:

$$\tilde{B}_{\mu\nu} = \frac{i}{2}\bar{M}\Gamma_{\mu\nu}M \tag{36}$$

[10] In order to implement the Faddeev-Popov procedure, we required that there are no *smooth* reducible connections on X. There is no contradiction here, since $\tilde{A}$ is singular on Σ.

where $\Gamma_{\mu\nu} = \frac{1}{2}[\Gamma_\mu, \Gamma_\nu]$ and the Γ's are the Clifford matrices. In the general case the solutions of our monopole equations (34) and (35) are only a subclass of Witten's. The situation here is similar to the comparison between a four-dimensional BF theory to a gravitational theory in the first order formalism. In order to connect the above two theories one has to require that the B field is the "square" of the vierbein [5]. In the situation described in this paper, in order to connect our BF theory with Witten monopole equation, one has to assume the validity of (36).

However, we want to stress that the results we present in the following of this paper are completely independent on the above relation between our and Witten's monopole equations.

Let us now look for an explicit solution of (34). Requiring that, in the limit $g \to 0$, we have non trivial solutions of (30) implies $\sin(\frac{g}{2}\phi_m) \neq 0$. In the path integral the effect of the cosmological-like term

$$\Delta S = -\frac{g^2}{4}\int_X \mathrm{Tr}\,(B \wedge B) \tag{37}$$

with $g \approx 0$ but $g \neq 0$ is to select the minimal energy solutions, obtained when $|\sin(\frac{g}{2}\phi_m)| = 1$, thus giving flux quantization.

We can choose coordinates (x, y, u, v) on X locally in a neighborhood of the base point $\bar{x}^\mu$ so that the surface Σ is given by the equations $x = y = 0$. For the Poincaré dual form ω restricted to Σ we can take [14]

$$\omega(x, y) = \psi(x, y)dx \wedge dy \tag{38}$$

where ψ is a bump function on $\mathbf{R}^2$, supported near $\vec{\bar{x}} = (x, y) = (0, 0)$ and with integral 1. We can choose $\psi(x, y) = \delta^{(2)}(x, y)$, with $\delta^{(2)}(x, y)$ the two-dimensional delta function. Therefore we may think that the Maxwell's field strength f is not zero only on Σ by choosing $\alpha = \alpha_\mu dx^\mu = \alpha_1 dx + \alpha_2 dy$, i.e. $\alpha_3 = \alpha_4 = 0$, so that $f = d\alpha = f_{12} dx \wedge dy$, where $f_{12} \equiv (\partial_1\alpha_2 - \partial_2\alpha_1)$, $\partial_i \equiv \partial/\partial x^i$ $(i = 1, 2)$, $\vec{x} \equiv \{x^i\} \equiv (x, y)$. Then (34) through (38) becomes at order g^2

$$f_{12}(x, y) = \mp 2\pi q_m g \delta^{(2)}(x, y) \tag{39}$$

The Maxwell potential α_i which solves (39) can be written as[11]

$$\alpha_i(\vec{x}) = \pm q_m g(\epsilon_{ij} + i\delta_{ij})\partial^j \log|\vec{x} - \vec{\bar{x}}| \tag{40}$$

This is the Kohno connection [23] that is obtained in a similar contest also in [24]. Indeed one has that

$$\partial_i\partial^i \log|\vec{x} - \vec{\bar{x}}| = 2\pi\delta^{(2)}(\vec{x} - \vec{\bar{x}}) \tag{41}$$

and locally around $\vec{\bar{x}}$ the monopole magnetic flux of α/g is

$$\phi_m = \pm 2\pi q_m Q(\Sigma, \Sigma) \tag{42}$$

[11] Dropping the assumption of minimal energy, the general solution is $\alpha_i(\vec{x}) = cg(\epsilon_{ij} + i\delta_{ij})\partial^j \log|\vec{x} - \vec{\bar{x}}|$ with $c/q_m = \sin(\pi gc)$.

where

$$Q(\Sigma,\Sigma') = \int_X \omega[\Sigma]\wedge\omega[\Sigma'] \tag{43}$$

denotes the *algebraic intersection number* [14] of the oriented surfaces Σ and Σ'. Notice that (43) is well defined even when $\Sigma' = \Sigma$. For the minimal energy solution one has $\sin(g\pi q_m) = 1$, implying

$$2gq_m Q(\Sigma,\Sigma) \equiv 1 \pmod 4 \tag{44}$$

which corresponds to the *Dirac quantization condition*. From (42) it follows that the operator (21), when expressed in terms of the fields $(\tilde{A},\,\tilde{B})$ is related to the exponential of the "magnetic flux".

Summarizing, we have shown that the inclusion of the gauge-invariant non-local operator $M(\Sigma)$ into the expectation value (15) given by the non-abelian SU(2) BF^+-theory reduces this theory to an effective abelian one with monopoles and *without* the $M(\Sigma)$ operator. The resulting quantum field theory, which is "dual" to the original one, is coupled to monopoles and is written in terms of an abelian connection α on a holomorphic line bundle L_Σ (the monopole line bundle), and a self-dual two-form $\tilde{B}$. This result proves (17) in the semiclassical approximation.

At this perturbative order the underlying moduli space is the classical one and coincides with the moduli space $\tilde{\mathcal{M}}$.

Thus the partition function given by the path-integral written in the r.h.s. of (17) (or equivalently the "one-point" function described by the l.h.s. of (17)) is equal to the number of points in our moduli-space of the fields $(\tilde{A},\tilde{B})$, counted with a $\pm$ sign

$$\langle 1\rangle_{(\tilde{\mathcal{M}},L_\Sigma)} \equiv \#\tilde{\mathcal{M}}(\tilde{A},\tilde{B};X,L_\Sigma) \quad . \tag{45}$$

Equation (45) is the same as in Witten [7, 18]. Equation (45) follows from the fact that the integration over $\tilde{B}$, gives a delta-functional contribution, i.e. $\delta(d^+\tilde{A}+2\pi q_m g\omega \mathrm{Tr}\{\mathrm{Hol}(C)t^3\})$, which in turn gives a counting measure (with signs) on the space $\mathcal{M}$.

Observe that the $\pm$ signs are provided by the ratio between the functional determinant coming from the ghosts[12] and the Jacobian associated with the delta-functional.

6 Donaldson's polynomial invariants

In this section we show how the original Donaldson polynomial invariants [14, 17] naturally come in the framework of our theory after the introduction of Wilson-line operators

$$W(C) = \mathrm{Tr}\, P\exp(\oint_C A) \tag{46}$$

[12] The gauge fixing (12) requires by (13) a Faddeev–Popov term $\bar{q}\wedge d_A q$, where $\bar{q}$ is the anti-self-dual anti-ghost associated to the 1-form ghost q.

Let us now consider a closed oriented surface $\hat{\Sigma}$. We open a small contractible disk Σ'' so that $\hat{\Sigma} = \Sigma' \cup \Sigma''$ and compute the following expectation value:

$$\langle M(\Sigma) W(C) \rangle \tag{47}$$

where $C = \partial\Sigma' = -\partial\Sigma''$. Since the Wilson loop does not depend on the B-field, we can perform the B integration exactly as in the previous section. Then the resulting delta-functional restricts the A-integration to the reducible connections $\tilde{A}$ and this allows to write the Wilson loop as

$$W(C) = \text{ Tr } \exp(\int_{\Sigma'} f\, t_3) \tag{48}$$

Since in this case one has $f = \mp 2\pi q_m g \omega[\Sigma]$ and $2 g q_m Q(\Sigma, \Sigma) = 1 + 4n$, (47) gives

$$\langle M(\Sigma) W(C) \rangle = \langle M(\Sigma) \rangle\, 2 \cos \left[\frac{\pi}{2} (1 + 4n) \frac{Q(\Sigma, \hat{\Sigma})}{Q(\Sigma, \Sigma)} \right] \tag{49}$$

If the monopole has the fundamental charge ($n = 0$), (49) gives the algebraic intersection number $Q(\Sigma, \hat{\Sigma})$ modulo $2Q(\Sigma, \Sigma)$. Notice that the insertion of the operator $M(\Sigma)$ creates a monopole whose flux through Σ' measures the intersection number.

We now move to the general case of $d + \hat{d}$ classes $[\Sigma_1], \ldots, [\Sigma_d]$ and $[\hat{\Sigma}_1], \ldots, [\hat{\Sigma}_{\hat{d}}]$ in $H_2(X, \mathbf{Z})$. We can associate as before $\hat{d}$ loops $C_1, \ldots, C_{\hat{d}}$ to the $[\hat{\Sigma}_b]$ and compute the expectation value

$$\left\langle \prod_{a=1}^{d} M(\Sigma_a) \prod_{b=1}^{\hat{d}} W(C_b) \right\rangle \tag{50}$$

which turns out to be proportional to

$$2^{\hat{d}} \prod_{a=1}^{d} \prod_{b=1}^{\hat{d}} \cos \left(\frac{\pi}{2} (1 + 4n_a) \frac{Q(\Sigma_a, \hat{\Sigma}_b)}{Q_0(\Sigma_.)} \right) \tag{51}$$

where $Q_0(\Sigma_.) = \text{G.C.D.}\{Q(\Sigma_a, \Sigma_a); a = 1, \ldots, d\}$.

Moreover, choosing $d = \hat{d}$ and $\Sigma_a = \hat{\Sigma}_a$, $a = 1, \ldots, d$, one gets a symmetrized version of (51), which is a generating functional for the Donaldson polynomial invariants associated to X, $q_X([\Sigma_1], \ldots, [\Sigma_d])$ [17].

From a physical point of view the Wilson-loops detect the monopole flux generated by the insertion of the operators $M(\Sigma)$ associated to surfaces. Monopoles of the same charge on the two sides of the surfaces give opposite contributions to the flux,[13] which turn out to be proportional to the algebraic intersection numbers.

[13] This picture, usually understood as a "double monopole layer", has been suggested to us by A. Sagnotti and has been firstly considered by A. Polyakov [22] in 3D-QED and more recently in Ref. [24] in 4D-QED.

7 Conclusions

In this work we have studied the self-dual BF theory as the first order version of pure Yang–Mills theory with theta-term. In this framework we have the Wilson loop operator $W(C)$, which is the "order operator" for ordinary Yang–Mills theory, and an operator $M(\Sigma)$ playing the rôle of a "disorder parameter" in the sense of Ref. [15]. Indeed the abelian reduction (31) maps the highly non-trivial relations among the "Weyl group operators"[14] Hol and M into the 't Hooft commutation relations.

From a physical point of view, the insertion of $M(\Sigma)$ plays the rôle of the standard "abelian projection" in pure QCD without Higgs mechanism [16].

The expectation values of $M(\Sigma)$ alone and of the combination $M(\Sigma)W(C)$ are related to the Donaldson–Witten invariants of a Kähler manifold X [14, 18, 17, 7, 20, 21]. However, these invariants arise here in a different setting with respect to the one considered by Witten.

Yet the computation of the invariants relies on the existence of monopole equations both here and in Witten's approach.

One of the differences between our framework and Witten's one, is that our observables introduce singularities, so our theory appears to be non-trivial, even if we work with $\mathbf{R}^4$. For a related approach see also [26].

The observable $M(\Sigma)$ (in the case when Σ is a torus) can be given a geometrical interpretation as parallel transport of one of the fundamental cycles along the other. This suggests that BF theory can be described as a "gauge theory of loops" [13]. Our formulation gives an effective field-theoretical description of the string picture for QCD introduced in Ref. [25].

All our calculations have been done in the limit $g \to 0$. In this limit both the topological BF theory with the "cosmological term" (11) in 4-dimensions and the Yang–Mills theory with theta-term appear to detect 4-manifolds invariants. For these two theories to be equivalent (on shell), the choice of the "gauge" $B^- = 0$ (that breaks the larger "topological invariance" of (11)) is a key ingredient.

If we instead consider a "pure" BF theory (i.e. we take $g = 0$) and do not impose the constraint of self-duality $B^- = 0$, we can study a different topological field theory, as discussed in [9, 11, 12]. Perturbation theory in $\kappa = q_m g$ detects 2-knots, i.e. embedded (or immersed) 2-surfaces, up to diffeomorphisms of the 4-manifolds.

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[14] Notice that by (11) in $3+1$ dimensions the spatial components $\epsilon_{ijk}B_{jk}$ are canonically conjugated to A_i.

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